

Intermediate-statistics spin waves

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Abstract: In this paper, we show that spin waves, the elementary excitation of the Heisenberg magnetic system, obey a kind of intermediate statistics with a finite maximum occupation number n . We construct an operator realization for the intermediate statistics obeyed by magnons, the quantized spin waves, and then construct a corresponding intermediate-statistics realization for the angular momentum algebra in terms of the creation and annihilation operators of the magnons. In other words, instead of the Holstein-Primakoff representation, a bosonic representation subject to a constraint on the occupation number, we present an intermediate-statistics representation with no constraints. In this realization, the maximum occupation number is naturally embodied in the commutation relation of creation and annihilation operators, while the Holstein-Primakoff representation is a bosonic operator relation with an additional putting-in-by-hand restriction on the occupation number. We deduce the intermediate-statistics distribution function for magnons from the intermediate-statistics commutation relation of the creation and annihilation operators directly, which is a modified Bose-Einstein distribution. Based on these results, we calculate the dispersion relations for ferromagnetic and antiferromagnetic spin waves. The relations between the intermediate statistics that magnons obey and the other two important kinds of intermediate statistics, Haldane-Wu statistics and the fractional statistics of anyons, are discussed. We also compare the spectrum of the intermediate-statistics spin wave with the exact solution of the one-dimensional $s = 1/2$ Heisenberg model, which is obtained by the Bethe ansatz method. For ferromagnets, we take the contributions from the interaction between magnons (the quartic contribution), the next-to-nearest-neighbor interaction, and the dipolar interaction into account for comparison with the experiment.

Keywords: Spin chains, ladders and planes (Theory)

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1 Introduction

At low temperatures, the elementary excitations of a magnetic system, a periodic system of localized spins coupled by exchange interaction, are magnons, the quantized spin waves. In this paper, we show that magnons obey a kind of intermediate statistics in which the

maximum number of particles in any quantum state is neither 1 nor ∞ , but equals a finite number n . That is to say, magnons are intermediate-statistics type quasiparticles.

Let us first recall the common treatment of a magnetic system. Take the ferromagnet as an example. The Hamiltonian of the Heisenberg model of a ferromagnetic system describing the exchange interaction between neighboring spins reads

$$H = \sum_{mn} J_{mn} \mathbf{S}_m \cdot \mathbf{S}_n, \quad (1)$$

where J_{mn} is the exchange coefficient and \mathbf{S}_m and \mathbf{S}_n are the spins at m -th and n -th sites. For such a spin system, introducing the spin deviation operator at site ℓ ,

$$N_\ell = S - S_\ell^z, \quad (2)$$

where S_ℓ^z is the z -component of the spin operator \mathbf{S}_ℓ and $\mathbf{S}_\ell^2 = S(S+1)$, we have

$$\begin{aligned} S_\ell^+ |N_\ell\rangle &= \sqrt{2S - (N_\ell - 1)} \sqrt{N_\ell} |N_\ell - 1\rangle, \\ S_\ell^- |N_\ell\rangle &= \sqrt{N_\ell + 1} \sqrt{2S - N_\ell} |N_\ell + 1\rangle, \\ S_\ell^z |N_\ell\rangle &= S - N_\ell, \end{aligned} \quad (3)$$

where $|N_\ell\rangle$ is the eigenstate of the spin deviation operator, i.e., $N_\ell |N_\ell\rangle = N_\ell |N_\ell\rangle$, $[S_\ell^+, S_\ell^-] = 2S_\ell^z \delta_{\ell\ell'}$, and $[S_\ell^z, S_\ell^\pm] = \pm S_\ell^\pm \delta_{\ell\ell'}$ [1]. A natural restriction is

$$0 \leq N_\ell \leq 2S, \quad (4)$$

since S_ℓ^z must be less than the magnitude of \mathbf{S}_ℓ . The relation (3) leads to an operator realization for the angular momentum:

$$\begin{aligned} S_\ell^+ &= \sqrt{2S - N_\ell} a_\ell^{Bose}, \\ S_\ell^- &= a_\ell^{Bose\dagger} \sqrt{2S - N_\ell}, \\ S_\ell^z &= S - N_\ell, \end{aligned} \quad (5)$$

where a_ℓ^{Bose} , $a_\ell^{Bose\dagger}$, and N_ℓ^{Bose} satisfy the bosonic commutation relation:

$$\begin{aligned} [a_\ell^{Bose}, a_{\ell'}^{Bose\dagger}] &= \delta_{\ell\ell'}, \\ [N_\ell, a_{\ell'}^{Bose\dagger}] &= a_\ell^{Bose\dagger} \delta_{\ell\ell'} \text{ and } [N_\ell, a_{\ell'}^{Bose}] = -a_\ell^{Bose} \delta_{\ell\ell'}. \end{aligned} \quad (6)$$

From equation (6), we can see that $a_\ell^{Bose\dagger}$ creates and a_ℓ^{Bose} annihilates a localized spin deviation at a single site. This is the Holstein-Primakoff representation of angular momentum algebra [2].

It is known that the Holstein-Primakoff representation is not a genuine bosonic realization of angular momentum algebra, since though the operators satisfy the bosonic commutation relation (6), there still exists an additional restriction on the value of the spin deviation N_ℓ , equation (4). When $N_\ell > 2S$, the representation is not faithful, while

in the Bose-Einstein case, N_ℓ can take on any value. That is to say, the Holstein-Primakoff representation corresponds essentially to a kind of intermediate statistics with a maximum occupation number $2S$. In the Holstein-Primakoff representation, though there exists a maximum occupation number $n = 2S$, the maximum occupation number is not embodied in the operator relation (6); in fact, the Holstein-Primakoff representation is a bosonic realization with a putting-in-by-hand restriction on the occupation number. As a result, however, when using the Holstein-Primakoff representation to solve the spectrum, only the operator relation is used, so the influence of the restriction on the occupation number is ignored.

The above analysis shows that the spin deviation does not obey Bose-Einstein statistics, but obeys a kind of intermediate statistics with a finite maximum occupation number. In fact, as discussed in [3, 4], if one wants to construct an operator realization for the angular momentum algebra by a single set of creation and annihilation operators, he needs a kind of intermediate statistics rather than the Bose-Einstein or Fermi-Dirac case. Nevertheless, in the common treatment of spin waves, the spin deviations are regarded as bosons: the commutation relation of creation and annihilation operators is taken as equation (6) and the statistical distribution is taken as the Bose-Einstein distribution.

In this paper, we will construct an intermediate-statistics operator realization for the angular momentum algebra, in which the maximum occupation number is naturally embodied in the commutation relation of creation and annihilation operators. Therefore, all results based on this intermediate-statistics realization can naturally take the influence of the restriction of occupation number into account. As a comparison, the Holstein-Primakoff representation is a constrained bosonic representation, i.e., a bosonic realization subject to a constraint on the occupation number, so the influence of the maximum occupation number cannot be taken into account directly.

From the intermediate-statistics realization for the angular momentum algebra, we calculate the corresponding intermediate-statistics distribution function. We show that the statistical distribution that magnons obey is a modified Bose-Einstein distribution.

Based on the intermediate-statistics realization for the angular momentum algebra and the intermediate-statistics distribution function, we calculate the dispersion relations of spin waves for ferromagnet and antiferromagnet.

The magnons, as shown in the present paper, obey a kind of intermediate statistics. As comparisons, we will discuss the relations between the intermediate statistics that magnons obey and the two important kinds of intermediate statistics: Haldane-Wu fractional statistics [5] and the fractional statistics of anyons [6]. Haldane-Wu fractional statistics is constructed based on the generalization of the Pauli exclusion principle. The concept of anyons is introduced by analyzing the symmetry properties of the wavefunction of identical particles: the change of the phase factor of the wavefunction when two identical particles exchange, instead of $+1$ or -1 , is generalized to an arbitrary phase factor $e^{i\theta}$. Each of these two kinds of intermediate statistics has an intermediate-statistics

parameter: g (Haldane-Wu) and $\alpha = \theta/\pi$ (anyon). The roles of g and α in Haldane-Wu statistics and in the fractional statistics of anyons are just as the role of the parameter n in the intermediate statistics obeyed by magnons. In this paper, we will discuss the relations of g and n and α and n , respectively.

A special case of the Heisenberg model, the one-dimensional $s = 1/2$ Heisenberg spin chain, can be solved exactly by the Bethe ansatz [7, 8, 9]. By this exact solution, we can check the validity of our result directly. Concretely, we will compare the exact spectrum obtained by the Bethe ansatz method with the spectrum obtained by the intermediate-statistics method and the spectrum obtained by the Holstein-Primakoff method.

Moreover, we also compare our result with the experimental data of EuO given in [10]. The result shows that at low temperatures and low frequencies, the intermediate-statistics spin wave model is more accurate than the bosonic spin wave model.

The present paper discusses the intermediate statistics and the intermediate-statistics realization of angular momentum algebra and their applications to magnetic systems. On the one hand, the realization of angular momentum algebra has been of interest for a long time. Besides the well-known Schwinger representation [11] and the Holstein-Primakoff representation [2], there are many other schemes, including the realization of $su(2)$ algebra [3, 4, 12, 13, 14], and the realization of $su_q(2)$ and $su_q(n)$ algebra [15, 16]. On the other hand, the spin wave plays an important role in magnetic problems [17, 18]. The concept of spin wave has become a widely applied tool in the fields related to magnetism. It has been applied to study magnetic semiconductors [19], quasiequilibrium spin systems [20], ballistic thermal transport [21] and the thermodynamics [22] in the Heisenberg spin chain, one-dimensional ferromagnetic Bose gases [23], spin-wave excitations in cylindrical ferromagnetic nanotubes [24], etc. In experiment, the properties of spin wave have been directly measured [25, 26]. The realizations of angular momentum algebra are successful in describing magnetism in various quantum systems [27]. In the application of the realizations of angular momentum algebra to magnetism, the realization is either bosonic or fermionic. However, as shown above, magnons obey neither Bose-Einstein nor Fermi-Dirac statistics. Therefore, a kind of intermediate-statistics treatment is needed. As generalizations of Bose-Einstein and Fermi-Dirac statistics, many schemes of intermediate statistics have been discussed [5, 6, 28, 29, 30].

In section 2, we construct an intermediate-statistics operator realization for the angular momentum algebra. In section 3, we calculate the intermediate-statistics distribution function for magnons based on the commutation relation between creation and annihilation operators of magnons. In sections 4 and 5, we calculate the dispersion relations for ferromagnetic and antiferromagnetic spin waves. In section 6, we discuss the relation between the intermediate statistics that magnons obey and Haldane-Wu fractional statistics and the relation between intermediate statistics and the fractional statistics of anyons. In section 7, we compare our result with the exact solution obtained by the Bethe ansatz method. In section 8, we compare the dispersion relation of a ferromagnetic system, which

is calculated based on intermediate statistics, with the experimental data of *EuO*. The conclusions and discussions are given in section 9.

2 Intermediate-statistics operator realization for angular momentum algebra

For solving the spectrum of a magnetic system with Hamiltonian (1), we need a representation of angular momentum algebra. As discussed above, the Holstein-Primakoff representation is a bosonic realization with a putting-in-by-hand restriction on the occupation number, and when using it to solve the spectrum, since only the operator relation is taken into account, the information of the restriction on the occupation number is ignored. For taking the influence of the restriction on the occupation number into account, we need a representation in which the information of the maximum occupation number is embodied in the operator relation rather than put in an additional restriction by hand. In this section, we construct an intermediate-statistics operator realization for the angular momentum algebra, in which the maximum occupation number is naturally embodied in the commutation relation of creation and annihilation operators.

In the case of the spin wave, we only focus on the low-lying excitation. In other words, in our case the spin deviation N_ℓ is always very small, and then equation (3) can be expanded around $N_\ell = 0$; only taking the next-to-leading-order contribution into account, we have

$$\begin{aligned} S_\ell^+ |N_\ell\rangle &= \sqrt{2S} \left(1 - \frac{N_\ell - 1}{4S}\right) \sqrt{N_\ell} |N_\ell - 1\rangle, \\ S_\ell^- |N_\ell\rangle &= \sqrt{2S} \sqrt{N_\ell + 1} \left(1 - \frac{N_\ell}{4S}\right) |N_\ell + 1\rangle, \\ S_\ell^z |N_\ell\rangle &= S - N_\ell. \end{aligned} \quad (7)$$

For constructing a realization of angular momentum algebra, we first introduce an intermediate-statistics operator realization:

$$\begin{aligned} [a_\ell, a_{\ell'}^\dagger] &= \left\{ \frac{1 - \frac{N_\ell}{n}}{\left(1 - \frac{N_\ell}{2n}\right)^2} + N_\ell \left[\frac{1 - \frac{N_\ell}{n}}{\left(1 - \frac{N_\ell}{2n}\right)^2} - \frac{1 - \frac{N_\ell - 1}{n}}{\left(1 - \frac{N_\ell - 1}{2n}\right)^2} \right] \right\} \delta_{\ell\ell'}, \\ [N_\ell, a_{\ell'}^\dagger] &= a_{\ell'}^\dagger \delta_{\ell\ell'} \text{ and } [N_\ell, a_{\ell'}] = -a_{\ell'} \delta_{\ell\ell'}, \end{aligned} \quad (8)$$

where $n = 2S$. It can be directly checked that such an operator realization corresponds to a kind of intermediate statistics with a maximum occupation number n .

By creation, annihilation, and number operators, a_ℓ , a_ℓ^\dagger , and N_ℓ , we can construct an

intermediate-statistics realization of angular momentum algebra:

$$\begin{aligned} S_\ell^+ &= \sqrt{2S} \left(1 - \frac{N_\ell}{4S}\right) a_\ell, \\ S_\ell^- &= \sqrt{2S} a_\ell^\dagger \left(1 - \frac{N_\ell}{4S}\right), \\ S_\ell^z &= S - N_\ell. \end{aligned} \quad (9)$$

It can be directly checked that, with the commutation relation (8), S_ℓ^+ , S_ℓ^- , and S_ℓ^z satisfy the operator relation of angular momentum:

$$\begin{aligned} [S_\ell^z, S_{\ell'}^\pm] &= \pm S_\ell^\pm \delta_{\ell\ell'}, \\ [S_\ell^+, S_{\ell'}^-] &= 2S_\ell^z \delta_{\ell\ell'}. \end{aligned} \quad (10)$$

In this scheme, a_ℓ^\dagger and a_ℓ are the creation and annihilation operators of a localized spin deviation at a single site, and N_ℓ is the spin deviation operator which can be expressed as

$$N_\ell = \frac{a_\ell^\dagger a_\ell + 2n \left[a_\ell^\dagger a_\ell + (n+1) - \sqrt{(n+1)^2 - (2n+1) a_\ell^\dagger a_\ell} \right]}{a_\ell^\dagger a_\ell + 4n}. \quad (11)$$

It should be emphasized that in such intermediate statistics, $N_\ell \neq a_\ell^\dagger a_\ell$. Concretely, for a given set of creation and annihilation operators, a_ℓ^\dagger and a_ℓ , the corresponding number operator N_ℓ can be constructed from the operator relations of a_ℓ^\dagger , a_ℓ , and N_ℓ . As shown in [3] and [4], the number operator of intermediate statistics in general cannot take the form of $N_\ell = a_\ell^\dagger a_\ell$. The realization of the number operator (11) shows that the spin deviation corresponds to a kind of intermediate statistics.

For describing the nonlocalized excitations of such a magnetic system, taking translational symmetry into account, we replace the creation (annihilation) operator a_ℓ^\dagger (a_ℓ) which creates (annihilates) localized spin deviations with the creation (annihilation) operator $b_{\mathbf{k}}^\dagger$ ($b_{\mathbf{k}}$) which creates (annihilates) nonlocalized excitations by the transformation

$$\begin{aligned} a_\ell &= \frac{1}{\sqrt{W}} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \boldsymbol{\ell}} b_{\mathbf{k}}, \\ a_\ell^\dagger &= \frac{1}{\sqrt{W}} \sum_{\mathbf{k}} e^{-i\mathbf{k} \cdot \boldsymbol{\ell}} b_{\mathbf{k}}^\dagger, \end{aligned} \quad (12)$$

where W is the number of lattice sites. Then

$$\begin{aligned} [b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] &= \left\{ \frac{1 - \frac{N_{\mathbf{k}}}{n}}{\left(1 - \frac{N_{\mathbf{k}}}{2n}\right)^2} + N_{\mathbf{k}} \left[\frac{1 - \frac{N_{\mathbf{k}}}{n}}{\left(1 - \frac{N_{\mathbf{k}}}{2n}\right)^2} - \frac{1 - \frac{N_{\mathbf{k}}-1}{n}}{\left(1 - \frac{N_{\mathbf{k}}-1}{2n}\right)^2} \right] \right\} \delta_{\mathbf{k}\mathbf{k}'}, \\ [N_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] &= b_{\mathbf{k}}^\dagger \delta_{\mathbf{k}\mathbf{k}'} \text{ and } [N_{\mathbf{k}}, b_{\mathbf{k}'}] = -b_{\mathbf{k}} \delta_{\mathbf{k}\mathbf{k}'}. \end{aligned} \quad (13)$$

$N_{\mathbf{k}}$, $b_{\mathbf{k}}^\dagger$, and $b_{\mathbf{k}}$ are the number, creation, and annihilation operators of magnons, respectively, which describe the nonlocalized elementary excitation of the system.

The above result shows that magnons, the elementary excitation of a magnetic system, obey the intermediate statistics defined by (13), which has a maximum occupation number n . Only when $n \rightarrow \infty$, such intermediate statistics returns to Bose-Einstein statistics. In other words, the spin wave is essentially an intermediate-statistics type elementary excitation.

3 The intermediate-statistics distribution function

When calculating the spectrum of a magnetic system, one needs to use the statistical distribution function of the magnon. In the common treatment, the statistical distribution is approximately taken as the Bose-Einstein distribution. However, as discussed above, the magnon in fact obeys intermediate statistics. In this section, we seek for the statistical distribution for an ideal magnon gas based on the commutation relation (13).

The average particle number of \mathbf{k} state can be obtained by

$$\langle N_{\mathbf{k}} \rangle = \frac{1}{Z} \text{Tr} \left[e^{-\beta(H-\mu N)} N_{\mathbf{k}} \right], \quad (14)$$

where $Z = \text{Tr} e^{-\beta(H-\mu N)}$, N is the total particle number operator, and μ is the chemical potential. For the low-lying excitation, we can expand the expression of the particle number operator, similar to equation (11), as

$$N_{\mathbf{k}} \simeq \frac{(2n+1)^2}{4n(n+1)} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{(2n+1)^3}{16n^2(n+1)^3} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right)^2. \quad (15)$$

By the relation

$$e^{-\beta(H-\mu N)} b_{\mathbf{k}}^{\dagger} = e^{-\beta(\varepsilon_{\mathbf{k}}-\mu)} b_{\mathbf{k}}^{\dagger} e^{-\beta(H-\mu N)}, \quad (16)$$

from (14), we achieve

$$\begin{aligned} \langle N_{\mathbf{k}} \rangle &= e^{-\beta(\varepsilon_{\mathbf{k}}-\mu)} \frac{1}{Z} \frac{(2n+1)^2}{4n(n+1)} \\ &\times \left\{ \text{Tr} \left[e^{-\beta(H-\mu N)} b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right] - \frac{2n+1}{4n(n+1)^2} \text{Tr} \left[e^{-\beta(H-\mu N)} b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right] \right\}, \end{aligned} \quad (17)$$

where $\varepsilon_{\mathbf{k}}$ is the energy of \mathbf{k} state. Based on the operator relation obtained in the above section, we construct

$$b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} = N_{\mathbf{k}} \frac{1 - \frac{N_{\mathbf{k}}-1}{n}}{\left(1 - \frac{N_{\mathbf{k}}-1}{2n}\right)^2} \quad \text{and} \quad b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} = (N_{\mathbf{k}}+1) \frac{1 - \frac{N_{\mathbf{k}}}{n}}{\left(1 - \frac{N_{\mathbf{k}}}{2n}\right)^2}. \quad (18)$$

Ignoring $\langle N_{\mathbf{k}}^2 \rangle$, we can solve $\langle N_{\mathbf{k}} \rangle$:

$$\langle N_{\mathbf{k}} \rangle = \frac{1 - \frac{4n^3+4n^2+2n+1}{16n^2(n+1)^3}}{e^{\beta(\varepsilon_{\mathbf{k}}-\mu)} - \left[1 - \frac{6n^3+8n^2+4n+1}{8n^2(n+1)^3} \right]}. \quad (19)$$

It can be directly seen from this equation that the statistical distribution defined by the commutation relation (13) is a modified Bose-Einstein distribution, and when $n \rightarrow \infty$, equation (19) returns to the Bose-Einstein distribution. In other words, the spin wave obeys a modified Bose-Einstein distribution.

4 Intermediate-statistics ferromagnetic spin waves

In this section, we calculate the dispersion relation of a ferromagnetic spin wave. The Hamiltonian of the Heisenberg model reads

$$H = -J_1 \sum_{\ell, \delta_1} \mathbf{S}_\ell \cdot \mathbf{S}_{\ell+\delta_1} - J_2 \sum_{\ell, \delta_2} \mathbf{S}_\ell \cdot \mathbf{S}_{\ell+\delta_2} + \cdots, \quad (20)$$

where δ_1 and δ_2 connect spin ℓ with its nearest and next-to-nearest neighbors and J_1 and J_2 denote the exchange parameters corresponding to the nearest-neighbor and next-to-nearest-neighbor couplings.

We first consider the nearest-neighbor contribution. The Hamiltonian reads

$$H = -J_1 \sum_{\ell, \delta_1} \left[S_\ell^z S_{\ell+\delta_1}^z + \frac{1}{2} \left(S_\ell^+ S_{\ell+\delta_1}^- + S_\ell^- S_{\ell+\delta_1}^+ \right) \right]. \quad (21)$$

Substituting the intermediate-statistics representation of angular momentum algebra (9) into (21), we have

$$H = H_0 + H_2 + H_4 \quad (22)$$

with

$$H_0 = -J_1 \frac{n^2}{4} \sum_{\ell, \delta_1} 1 = -J_1 W Z_1 \frac{n^2}{4}, \quad (23)$$

$$H_2 = J_1 \sum_{\ell, \delta_1} \left[n N_\ell - \frac{n}{2} \left(a_\ell a_{\ell+\delta_1}^\dagger + a_\ell^\dagger a_{\ell+\delta_1} \right) \right], \quad (24)$$

and

$$H_4 = -J_1 \sum_{\ell, \delta_1} \left[N_\ell N_{\ell+\delta_1} - \frac{1}{4} \left(a_\ell a_{\ell+\delta_1}^\dagger N_{\ell+\delta_1} + N_\ell a_\ell a_{\ell+\delta_1}^\dagger + a_\ell^\dagger N_{\ell+\delta_1} a_{\ell+\delta_1} + a_\ell^\dagger N_\ell a_{\ell+\delta_1} \right) \right], \quad (25)$$

where Z_1 is the number of nearest neighbors.

In the following, by the above operator relations and the statistical distribution function given in sections 2 and 3, we can calculate the spectrum directly.

4.1 The quadratic contribution

First, we calculate the contribution from the terms quadratic in the creation and annihilation operators. Substituting (12) into (24) gives

$$H_2 = J_1 Z_1 n \sum_{\mathbf{k}} N_{\mathbf{k}} - J_1 Z_1 \frac{n}{2} \sum_{\mathbf{k}} \gamma_{\mathbf{k}}^N \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \right). \quad (26)$$

Here $\gamma_{\mathbf{k}}^N$ is defined to be $\gamma_{\mathbf{k}}^N = \frac{1}{Z_1} \sum_{\delta_1} e^{i\mathbf{k} \cdot \delta_1}$ with $\gamma_{\mathbf{k}}^N = \gamma_{-\mathbf{k}}^N$ due to the symmetry. Then, substituting (18) into (26) and preserving only the first-order contribution of $N_{\mathbf{k}}$ gives

$$H_2 = J_1 Z_1 n \sum_{\mathbf{k}} \left[(1 - \gamma_{\mathbf{k}}^N) + \frac{1}{2(2n+1)^2} \gamma_{\mathbf{k}}^N \right] N_{\mathbf{k}}. \quad (27)$$

We achieve the dispersion relation of magnons to the second order:

$$\hbar\omega_{\mathbf{k}}^{(2)} = J_1 Z_1 n \left[(1 - \gamma_{\mathbf{k}}^N) + \frac{1}{2(2n+1)^2} \gamma_{\mathbf{k}}^N \right]. \quad (28)$$

The second term in (28) is the modification coming from the influence of the restriction on the occupation number.

4.2 The quartic contribution: the interaction between magnons

The contribution from the terms quartic in the creation and annihilation operators describes the interaction between magnons. A similar treatment can also be used to deal with the quartic terms.

From equation (25), by the operator relations given above, up to quartic terms, we obtain

$$\begin{aligned} H_4 = & -J_1 \frac{Z_1}{W} \sum_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4} \left\{ \left[1 + \frac{1}{4n(n+1)} \right]^2 \delta_{\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3, \mathbf{k}_4} \gamma_{\mathbf{k}_3 - \mathbf{k}_4}^N b_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_2} b_{\mathbf{k}_3}^\dagger b_{\mathbf{k}_4} \right. \\ & - \frac{1}{4} \left[1 + \frac{1}{4n(n+1)} \right] \left(\delta_{\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3, -\mathbf{k}_4} \gamma_{\mathbf{k}_1}^N b_{\mathbf{k}_1} b_{\mathbf{k}_2}^\dagger b_{\mathbf{k}_3}^\dagger b_{\mathbf{k}_4} + \delta_{\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3, -\mathbf{k}_4} \gamma_{\mathbf{k}_4}^N b_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_2} b_{\mathbf{k}_3} b_{\mathbf{k}_4}^\dagger \right. \\ & \left. \left. + \delta_{\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3, \mathbf{k}_4} \gamma_{\mathbf{k}_1}^N b_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_2}^\dagger b_{\mathbf{k}_3} b_{\mathbf{k}_4} + \delta_{\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3, \mathbf{k}_4} \gamma_{\mathbf{k}_4}^N b_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_2}^\dagger b_{\mathbf{k}_3} b_{\mathbf{k}_4} \right) \right\}. \quad (29) \end{aligned}$$

For long-wavelength spin waves, the main contribution comes from the interactions that do not change the state of the spin wave. For example, for the term in proportion to $b_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_2}^\dagger b_{\mathbf{k}_3} b_{\mathbf{k}_4}$, only the contributions corresponding to $\mathbf{k}_1 = \mathbf{k}_3$ and $\mathbf{k}_2 = \mathbf{k}_4$, or $\mathbf{k}_1 = \mathbf{k}_4$ and $\mathbf{k}_2 = \mathbf{k}_3$ remain [9]. Only taking these contributions into account, we approximately achieve

$$\begin{aligned} H_4 \simeq & -J_1 \frac{Z_1}{W} \sum_{\mathbf{k}_1 \mathbf{k}_2} \left\{ \left[1 + \frac{1}{4n(n+1)} \right]^2 \left(\gamma_0^N b_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_1} b_{\mathbf{k}_2}^\dagger b_{\mathbf{k}_2} + \gamma_{\mathbf{k}_2 - \mathbf{k}_1}^N b_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_1} b_{\mathbf{k}_2} b_{\mathbf{k}_2}^\dagger \right) \right. \\ & \left. - \frac{1}{4} \left[1 + \frac{1}{4n(n+1)} \right] \left[\gamma_{\mathbf{k}_1}^N b_{\mathbf{k}_1} b_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_2}^\dagger b_{\mathbf{k}_2} + 3\gamma_{\mathbf{k}_2}^N b_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_1} b_{\mathbf{k}_2} b_{\mathbf{k}_2}^\dagger + (3\gamma_{\mathbf{k}_1}^N + \gamma_{\mathbf{k}_2}^N) b_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_1} b_{\mathbf{k}_2}^\dagger b_{\mathbf{k}_2} \right] \right\}. \quad (30) \end{aligned}$$

By equation (18), up to $N_{\mathbf{k}}^2$, we obtain

$$H = -J_1 \frac{Z_1}{W} \sum_{\mathbf{k}_1 \mathbf{k}_2} \left[(1 - \gamma_{\mathbf{k}_1}^N) (1 - \gamma_{\mathbf{k}_2}^N) \right. \quad (31)$$

$$\left. + \frac{\gamma_{\mathbf{k}_1}^N + \gamma_{\mathbf{k}_2}^N + 2\gamma_{\mathbf{k}_1}^N \gamma_{\mathbf{k}_2}^N}{2(2n+1)^2} + \frac{\gamma_{\mathbf{k}_1}^N \gamma_{\mathbf{k}_2}^N}{4n(n+1)(2n+1)^2} \right] N_{\mathbf{k}_1} N_{\mathbf{k}_2}. \quad (32)$$

In the calculation, the relations $\gamma_0^N = 1$, $\gamma_{\mathbf{k}_2 - \mathbf{k}_1}^N = \gamma_{\mathbf{k}_2}^N \gamma_{\mathbf{k}_1}^N$ [31], $\sum_{\mathbf{k}} \gamma_{\mathbf{k}}^N = 0$, and $\sum_{\mathbf{k}_1 \mathbf{k}_2} \gamma_{\mathbf{k}_1}^N N_{\mathbf{k}_1} N_{\mathbf{k}_2} = \sum_{\mathbf{k}_1 \mathbf{k}_2} \gamma_{\mathbf{k}_2}^N N_{\mathbf{k}_1} N_{\mathbf{k}_2}$ have been used.

When the number of excited magnons fluctuates little, we can take the approximation [9]

$$N_{\mathbf{k}_1} N_{\mathbf{k}_2} \simeq \langle N_{\mathbf{k}_1} \rangle N_{\mathbf{k}_2} + N_{\mathbf{k}_1} \langle N_{\mathbf{k}_2} \rangle - \langle N_{\mathbf{k}_1} \rangle \langle N_{\mathbf{k}_2} \rangle. \quad (33)$$

Then

$$\begin{aligned} H_4 \simeq & -2J_1 \frac{Z_1}{W} \sum_{\mathbf{qk}} \left[(1 - \gamma_{\mathbf{q}}^N) (1 - \gamma_{\mathbf{k}}^N) + \frac{\gamma_{\mathbf{q}}^N + \gamma_{\mathbf{k}}^N + 2\gamma_{\mathbf{q}}^N \gamma_{\mathbf{k}}^N}{2(2n+1)^2} \right. \\ & \left. + \frac{\gamma_{\mathbf{q}}^N \gamma_{\mathbf{k}}^N}{4n(n+1)(2n+1)^2} \right] \langle N_{\mathbf{q}} \rangle N_{\mathbf{k}} + J_1 \frac{Z_1}{W} \sum_{\mathbf{k}_1 \mathbf{k}_2} \left[(1 - \gamma_{\mathbf{k}_1}^N) (1 - \gamma_{\mathbf{k}_2}^N) \right. \\ & \left. + \frac{\gamma_{\mathbf{k}_1}^N + \gamma_{\mathbf{k}_2}^N + 2\gamma_{\mathbf{k}_1}^N \gamma_{\mathbf{k}_2}^N}{2(2n+1)^2} + \frac{\gamma_{\mathbf{k}_1}^N \gamma_{\mathbf{k}_2}^N}{4n(n+1)(2n+1)^2} \right] \langle N_{\mathbf{k}_1} \rangle \langle N_{\mathbf{k}_2} \rangle. \end{aligned} \quad (34)$$

Consequently, the contribution from the interaction between magnons to the dispersion relation reads

$$\begin{aligned} \hbar\omega_{\mathbf{k}}^{(4)} = & -2J_1 \frac{Z_1}{W} \left\{ \sum_{\mathbf{q}} \left[(1 - \gamma_{\mathbf{q}}^N) + \frac{\gamma_{\mathbf{q}}^N}{2(2n+1)^2} \right] \langle N_{\mathbf{q}} \rangle \right. \\ & \left. - \gamma_{\mathbf{k}}^N \sum_{\mathbf{q}} \left[(1 - \gamma_{\mathbf{q}}^N) - \frac{1}{2(2n+1)^2} - \frac{\gamma_{\mathbf{q}}^N}{4n(n+1)} \right] \langle N_{\mathbf{q}} \rangle \right\}. \end{aligned} \quad (35)$$

It should be emphasized that the statistical distribution function $\langle N_{\mathbf{q}} \rangle$ is the intermediate-statistical distribution given by (19), rather than the Bose-Einstein distribution as that in the common treatment.

4.3 Comparison with the result of the Holstein-Primakoff representation

From the above discussion on the commutation relation of creation and annihilation operators, we have already known that magnons obey a kind of intermediate statistics with a maximum occupation number n , and when $n \rightarrow \infty$ this intermediate statistics returns to Bose-Einstein statistics. In the common treatment, one approximately assumes that magnons obey Bose-Einstein statistics. Under this approximation, the dispersion relation reads [9]

$$\hbar\omega_{\mathbf{k}} = J_1 Z_1 2S (1 - \gamma_{\mathbf{k}}^N) - 2J_1 \frac{Z_1}{W} (1 - \gamma_{\mathbf{k}}^N) \left[\sum_{\mathbf{q}} \langle N_{\mathbf{q}}^{Bose} \rangle - \sum_{\mathbf{q}} \gamma_{\mathbf{q}}^N \langle N_{\mathbf{q}}^{Bose} \rangle \right], \quad (36)$$

where $\langle N_{\mathbf{q}}^{Bose} \rangle$ denotes the Bose-Einstein distribution.

Comparing the dispersion relation (36) with the dispersion relation (28) and (35), we can see that when replacing Bose-Einstein statistics by intermediate statistics, some additional modifications relying on the maximum occupation number n appear. When the maximum occupation number $n \rightarrow \infty$, such modifications vanish.

Especially, the statistical distribution function $\langle N_{\mathbf{q}}^{Bose} \rangle$ in the Holstein-Primakoff result (36) is the Bose-Einstein distribution, while in the present result (35), the statistical distribution function $\langle N_{\mathbf{q}} \rangle$ is the intermediate-statistics distribution (19) which is a modified Bose-Einstein distribution.

4.4 Other contributions

For comparison with the experiment, we need to consider all effects as possible. In this subsection, we discuss the contribution from next-to-nearest neighbors and dipolar interactions.

4.4.1 The next-to-nearest-neighbor contribution

The contribution from the next-to-nearest-neighbor coupling can be calculated directly by the same procedure:

$$\begin{aligned} \hbar\omega_{\mathbf{k}}^{NN} = & J_2 Z_2 n \left[(1 - \gamma_{\mathbf{k}}^{NN}) + \frac{1}{2(2n+1)^2} \gamma_{\mathbf{k}}^{NN} \right] \\ & - 2J_2 \frac{Z_2}{W} \left\{ \sum_{\mathbf{q}} \left[(1 - \gamma_{\mathbf{q}}^{NN}) + \frac{\gamma_{\mathbf{q}}^{NN}}{2(2n+1)^2} \right] \langle N_{\mathbf{q}} \rangle \right. \\ & \left. - \gamma_{\mathbf{k}}^{NN} \sum_{\mathbf{q}} \left[(1 - \gamma_{\mathbf{q}}^{NN}) - \frac{1}{2(2n+1)^2} - \frac{\gamma_{\mathbf{q}}^{NN}}{4n(n+1)} \right] \langle N_{\mathbf{q}} \rangle \right\}. \end{aligned} \quad (37)$$

where Z_2 is the number of next-to-nearest neighbors and $\gamma_{\mathbf{k}}^{NN} = \frac{1}{Z_2} \sum_{\delta_2} e^{i\mathbf{k} \cdot \delta_2}$.

4.4.2 The dipolar interaction

In the above, we only take the contribution from the exchange coupling into account. Besides the exchange coupling, there still exists a classical dipolar interaction which is caused by the interaction between the magnetic moments. The Hamiltonian of the dipolar interaction reads

$$H_{dip} = \frac{1}{2} g^2 \mu_B^2 \sum_{i,j} \left[\frac{\mathbf{S}_i \cdot \mathbf{S}_j}{r_{ij}^3} - \frac{3(\mathbf{S}_i \cdot \mathbf{r}_{ij})(\mathbf{S}_j \cdot \mathbf{r}_{ij})}{r_{ij}^5} \right], \quad (38)$$

where μ_B is the Bohr magnon and $g = 2$ is the Landé factor. In principle, we need to substitute the intermediate-statistics operator realization of angular momentum algebra (9) and (8) into this Hamiltonian, and, then, calculate the influence of the intermediate-statistics dipolar interaction to the dispersion relation. However, since the contribution from the dipolar interaction, in comparison with the contribution from the exchange coupling, is small, we ignore the intermediate-statistics modification to the dipolar contribution, i.e., for the dipolar interaction, we approximately use the result obtained by the Holstein-Primakoff representation.

Under the assumption of isotropy, the contribution from the dipolar interaction, based on the result given in [32], can be approximately expressed as

$$\hbar\omega_{\mathbf{k}}^{dip} = \frac{4\pi}{3}g\mu_B M(T), \quad (39)$$

where $M(T)$ is the magnetization. Then the dispersion relation reads

$$\hbar\omega_{\mathbf{k}}^{total} = \hbar\omega_{\mathbf{k}} + \hbar\omega_{\mathbf{k}}^{dip}, \quad (40)$$

where $\hbar\omega_{\mathbf{k}}$ comes from the exchange coupling and $\hbar\omega_{\mathbf{k}}^{dip}$ comes from the dipolar interaction.

5 Intermediate-statistics antiferromagnetic spin waves

It has been shown that there do exist quantized spin waves in antiferromagnets [18]. In this section, we calculate the dispersion relation for antiferromagnetic spin waves based on the intermediate-statistics scheme.

For antiferromagnets, the spin structure of the crystal is considered as two interpenetrating sublattices A and B with the property that all nearest neighbors of a spin on A lie on B , and *vice versa*. The Hamiltonian reads

$$H = 2J \sum_{\ell\delta} \left[S_{A\ell}^z S_{B\ell+\delta}^z + \frac{1}{2} (S_{A\ell}^+ S_{B\ell+\delta}^- + S_{A\ell}^- S_{B\ell+\delta}^+) \right], \quad (41)$$

where ℓ runs over all sites of sublattice A . In this paper, we only consider the contribution from the nearest neighbors.

The antiferromagnetic ground state is approximately taken as the Néel state, in which the z component of each spin is S in sublattice A , and $-S$ in sublattice B .

The excited state of antiferromagnets can be treated by the similar treatment of ferromagnets. Similar to equation (9), introduce two sets of operator realizations for sublattice A and B , respectively:

$$\begin{aligned} S_{A\ell}^+ &= \sqrt{2S} \left(1 - \frac{N_\ell}{4S} \right) a_\ell, \\ S_{A\ell}^- &= \sqrt{2S} a_\ell^\dagger \left(1 - \frac{N_\ell}{4S} \right), \\ S_{A\ell}^z &= S - N_\ell, \end{aligned} \quad (42)$$

and

$$\begin{aligned} S_{B\ell}^+ &= \sqrt{2S} c_\ell^\dagger \left(1 - \frac{N_\ell}{4S} \right), \\ S_{B\ell}^- &= \sqrt{2S} \left(1 - \frac{N_\ell}{4S} \right) c_\ell, \\ S_{B\ell}^z &= -(S - N_\ell), \end{aligned} \quad (43)$$

where a_ℓ^\dagger , a_ℓ and c_ℓ^\dagger , c_ℓ are the creation and annihilation operators of the spin deviations on sublattice A and B , satisfying the operator relations (8) and (11), respectively.

Substituting (42) and (43) into Hamiltonian (41) gives

$$H = 2J \sum_{\ell\delta} \{ (S - N_\ell) (-S + N_{\ell+\delta}) + S \left[\left(1 - \frac{N_\ell}{4S} \right) a_\ell \left(1 - \frac{N_{\ell+\delta}}{4S} \right) c_{\ell+\delta} + a_\ell^\dagger \left(1 - \frac{N_\ell}{4S} \right) c_{\ell+\delta}^\dagger \left(1 - \frac{N_{\ell+\delta}}{4S} \right) \right] \}. \quad (44)$$

For low-lying excitations, we only take the contribution from the terms quadratic in the creation and annihilation operators into account:

$$H \simeq -2JWZS^2 + 2JS \sum_{\ell\delta} (N_\ell + N_{\ell+\delta}) + 2JS \sum_{\ell\delta} \left(a_\ell c_{\ell+\delta} + a_\ell^\dagger c_{\ell+\delta}^\dagger \right). \quad (45)$$

Here W denotes the number of the sites of sublattice A .

Introduce the transformations

$$a_\ell = \frac{1}{\sqrt{W}} \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\ell} b_{\mathbf{k}} \text{ and } a_\ell^\dagger = \frac{1}{\sqrt{W}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\ell} b_{\mathbf{k}}^\dagger, \quad (46)$$

and

$$c_\ell = \frac{1}{\sqrt{W}} \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\ell} d_{\mathbf{k}} \text{ and } c_\ell^\dagger = \frac{1}{\sqrt{W}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\ell} d_{\mathbf{k}}^\dagger. \quad (47)$$

Using the relation between N_ℓ and a_ℓ^\dagger , a_ℓ , c_ℓ^\dagger , and c_ℓ and substituting the above transformations into (45), up to the quadratic contribution, gives

$$H \simeq -2JWZS^2 + \frac{(2n+1)^2}{2n(n+1)} JSZ \sum_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + d_{\mathbf{k}}^\dagger d_{\mathbf{k}} \right) + 2JSZ \sum_{\mathbf{k}} \gamma_{\mathbf{k}} \left(b_{\mathbf{k}} d_{-\mathbf{k}} + b_{\mathbf{k}}^\dagger d_{-\mathbf{k}}^\dagger \right). \quad (48)$$

For diagonalizing the Hamiltonian (48), we introduce the Bogoliubov transformation which mixes the operators of the two sublattices:

$$\begin{aligned} b_{\mathbf{k}} &= u_{\mathbf{k}} \alpha_{\mathbf{k}} + v_{\mathbf{k}} \beta_{\mathbf{k}}^\dagger, & d_{-\mathbf{k}} &= u_{\mathbf{k}} \beta_{\mathbf{k}} + v_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger, \\ b_{\mathbf{k}}^\dagger &= u_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger + v_{\mathbf{k}} \beta_{\mathbf{k}}, & d_{-\mathbf{k}}^\dagger &= u_{\mathbf{k}} \beta_{\mathbf{k}}^\dagger + v_{\mathbf{k}} \alpha_{\mathbf{k}}. \end{aligned} \quad (49)$$

It can be checked directly that the Hamiltonian (48) can be diagonalized when $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are taken as

$$\begin{aligned} u_{\mathbf{k}}^2 &= \frac{1}{2} \left\{ \left[1 - \frac{16n^2 (n+1)^2}{(2n+1)^4} \gamma_{\mathbf{k}}^2 \right]^{-1/2} + 1 \right\}, \\ v_{\mathbf{k}}^2 &= \frac{1}{2} \left\{ \left[1 - \frac{16n^2 (n+1)^2}{(2n+1)^4} \gamma_{\mathbf{k}}^2 \right]^{-1/2} - 1 \right\}. \end{aligned} \quad (50)$$

Then, from equation (48), ignoring the high-order contribution, we achieve

$$H = -2JWZS^2 + 2JSZ \sum_{\mathbf{k}} \left\{ \left[\frac{(2n+1)^2}{4n(n+1)} u_{\mathbf{k}}^2 + \gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right] (\alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + \beta_{\mathbf{k}}^\dagger \beta_{\mathbf{k}}) + \left[\frac{(2n+1)^2}{4n(n+1)} v_{\mathbf{k}}^2 + \gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right] (\alpha_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger + \beta_{\mathbf{k}} \beta_{\mathbf{k}}^\dagger) \right\}. \quad (51)$$

By the operator relation

$$\alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} = N_{\alpha\mathbf{k}} \frac{1 - \frac{N_{\alpha\mathbf{k}}-1}{n}}{\left(1 - \frac{N_{\alpha\mathbf{k}}-1}{2n}\right)^2} \quad \text{and} \quad \alpha_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger = (N_{\alpha\mathbf{k}} + 1) \frac{1 - \frac{N_{\alpha\mathbf{k}}}{n}}{\left(1 - \frac{N_{\alpha\mathbf{k}}}{2n}\right)^2} \quad (52)$$

(the operator relation for $\beta_{\mathbf{k}}$ is the same as that of $\alpha_{\mathbf{k}}$), ignoring the high-order contribution, from (51), we achieve

$$H = -2JWZS^2 + 4JSZ \sum_{\mathbf{k}} \left[\frac{(2n+1)^2}{4n(n+1)} v_{\mathbf{k}}^2 + \gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right] + \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} (N_{\alpha\mathbf{k}} + N_{\beta\mathbf{k}}), \quad (53)$$

where the dispersion relation for antiferromagnetic magnons is

$$\hbar\omega_{\mathbf{k}} = 2JSZ \left[u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 + 2\gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} + \frac{1}{4n(n+1)} v_{\mathbf{k}}^2 - \frac{1}{(2n+1)^2} \gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right]. \quad (54)$$

Similar to the ferromagnetic case, the antiferromagnet magnon obeys intermediate statistics rather than Bose-Einstein statistics. When approximately regarding the magnon as bosons, based on the Holstein-Primakoff representation, the dispersion relation of the antiferromagnet magnon reads

$$\hbar\omega_{\mathbf{k}}^{HP} = 2JSZ (u_{\mathbf{k}}^{HP2} + v_{\mathbf{k}}^{HP2} + 2\gamma_{\mathbf{k}} u_{\mathbf{k}}^{HP} v_{\mathbf{k}}^{HP}), \quad (55)$$

where the superscript "HP" denotes that the corresponding result comes from the method of the Holstein-Primakoff representation. Comparing the dispersion relation (54) with (55), we can see that when the maximum occupation number $n \rightarrow \infty$, our result returns to the result of the Holstein-Primakoff representation which regards magnons as bosons. The magnitude of the modification relies on the value of n .

6 Comparison with other schemes of intermediate statistics

6.1 Comparison with Haldane-Wu fractional statistics

The above result shows that the magnons obey a kind of intermediate statistics with the statistical distribution (19). Let us compare this statistical distribution with another kind of intermediate statistics — Haldane-Wu fractional statistics [5]. The Haldane-Wu distribution function reads

$$\langle N_{\mathbf{k}}^{HW} \rangle = \frac{1}{\omega^{-1} + (g-1)}, \quad (56)$$

where ω is determined by $g \ln(1 - \omega) - \ln \omega = \beta(\varepsilon_{\mathbf{k}} - \mu)$.

In the Haldane-Wu distribution, there is an intermediate-statistics parameter g , and in the intermediate statistics obeyed by magnons, the intermediate-statistics parameter is n . A relation between these two intermediate-statistics parameters can be obtained by comparing the second virial coefficients. The second virial coefficient of a ν -dimensional ideal magnon gas with the dispersion relation $\varepsilon \propto p^s$ can be obtained directly:

$$a_2 = -\frac{1}{2^{\nu/s+1}} \frac{\nu \Gamma(\frac{\nu}{2})}{2 \Gamma(\frac{\nu}{s} + 1)} \frac{4n^3 + 8n^2 - 2}{4n^3 + 8n^2 + 2n - 1}. \quad (57)$$

The second virial coefficient of an ideal gas obeying Haldane-Wu fractional statistics reads $a_2^{HW} = -(1 - 2g)/2^{\nu/s+1}$ [33]. Comparing these two second virial coefficients gives

$$g = \frac{1}{2} \left[1 - \frac{\nu \Gamma(\frac{\nu}{2})}{2 \Gamma(\frac{\nu}{s} + 1)} \frac{4n^3 + 8n^2 - 2}{4n^3 + 8n^2 + 2n - 1} \right]. \quad (58)$$

6.2 Comparison with the fractional statistics of anyons

It is also interesting to compare this intermediate statistics with the fractional statistics of anyons, another scheme of intermediate statistics [6]. For the case of anyon, we of course only focus on two dimensions.

The concept of anyons is introduced by generalizing the change of the phase factor of a wavefunction when two identical particles exchange to an arbitrary phase factor $e^{i\theta}$. $\theta = 0$ and $\theta = \pi$ correspond to Bose-Einstein and Fermi-Dirac cases, respectively.

The second virial coefficient of an anyon gas reads [33, 34]

$$a_2 = -\frac{1}{4} (1 - 4\alpha + 2\alpha^2), \quad (59)$$

where $\alpha = \theta/\pi$. Comparing this result with the second virial coefficient (57) with $\nu = 2$ and $s = 2$ gives

$$\alpha = 1 - \sqrt{1 - \frac{2n+1}{8n^3 + 16n^2 + 4n - 2}}. \quad (60)$$

7 Comparison with the exact result of the Bethe ansatz method: the spectrum

In this section, we compare our result with the exact solution of the one-dimensional spin 1/2 Heisenberg model.

By the Bethe ansatz, one can find the exact solutions of certain one-dimensional quantum many-body models. Taking ferromagnets as an example, we compare our result given in section 4 with the exactly solved one-dimensional $s = 1/2$ Heisenberg model with two down spins.

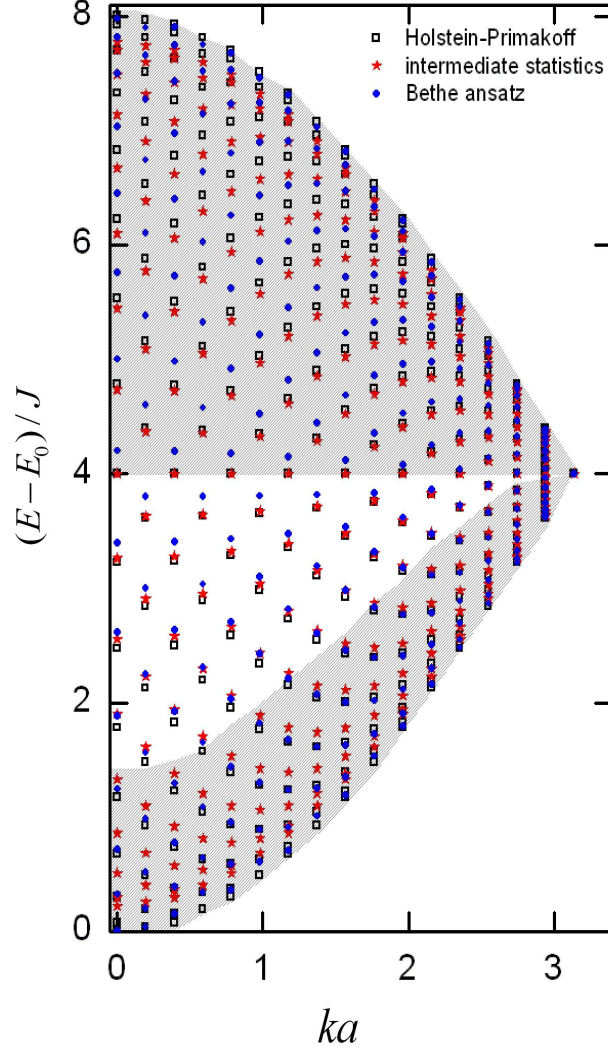


Figure 1: The spectra of the one-dimensional $s = 1/2$ Heisenberg chain given by the Bethe ansatz method (the exact solution), by the Holstein-Primakoff method, and by the intermediate-statistics method. In the unshaded area, in comparison with the exact result, the intermediate-statistics result is better than the result given by the Holstein-Primakoff method.

The exact spectrum of the one-dimensional $s = 1/2$ Heisenberg model with two down spins is given by [7, 8, 9]

$$E - E_0 = J(2 - \cos k_1 a - \cos k_2 a), \quad (61)$$

where a is the lattice constant and k_1, k_2 are determined by

$$\begin{aligned} Nk_1 a &= 2\pi\lambda_1 + \theta, \\ Nk_2 a &= 2\pi\lambda_2 - \theta, \\ 2 \cot \frac{\theta}{2} &= \cot \frac{k_1 a}{2} - \cot \frac{k_2 a}{2}, \end{aligned} \quad (62)$$

with $\lambda_1, \lambda_2 = 0, 1, 2, \dots, N-1$ and $\lambda_2 \geq \lambda_1$.

Moreover, our result of the spectrum for the corresponding case can be directly obtained by equation (28) with $n = 1$.

The spectra obtained by the Bethe ansatz (the exact one), by the Holstein-Primakoff method, and by the intermediate-statistics method are sketched in figure 1.

Comparing with the exact result obtained by the Bethe ansatz, we can see that in some cases (the unshadowed area) our result (the intermediate-statistics magnons) is more accurate than the standard Holstein-Primakoff result (the bosonic magnons).

8 Comparing with the experiment

From equation (40), we can obtain the relation between the spin-wave energies and the temperature by the self-consistent calculation. We will consider the spin-wave dispersion relation of EuO since the spin-wave dispersion for EuO is isotropic. The Eu^{2+} ions in EuO form simple fcc lattices, so the number of the nearest neighbors and next-to-nearest neighbors are $Z_1 = 12$ and $Z_2 = 6$, the exchange parameters J_1 to nearest neighbors and J_2 to next-to-nearest neighbors are $J_1 = 0.606k_B$ and $J_2 = 0.119k_B$ [35], where k_B is the Boltzmann constant, and $S = 7/2$.

The calculation results are plotted in figure 2. The experimental data are taken from [10].

In comparison with the experimental data, we can see that at low temperatures and low frequencies, the result of the intermediate-statistics spin waves is more accurate than the result of the bosonic spin waves, and at high temperatures and high frequencies, the result of bosonic spin waves is better.

9 Discussion and Conclusions

It is shown that magnons, the elementary excitation of a Heisenberg magnetic system, obey a kind of intermediate statistics with a maximum occupation number $n = 2S$. In the common treatment, the solution of the spectrum of a magnetic system is based on the

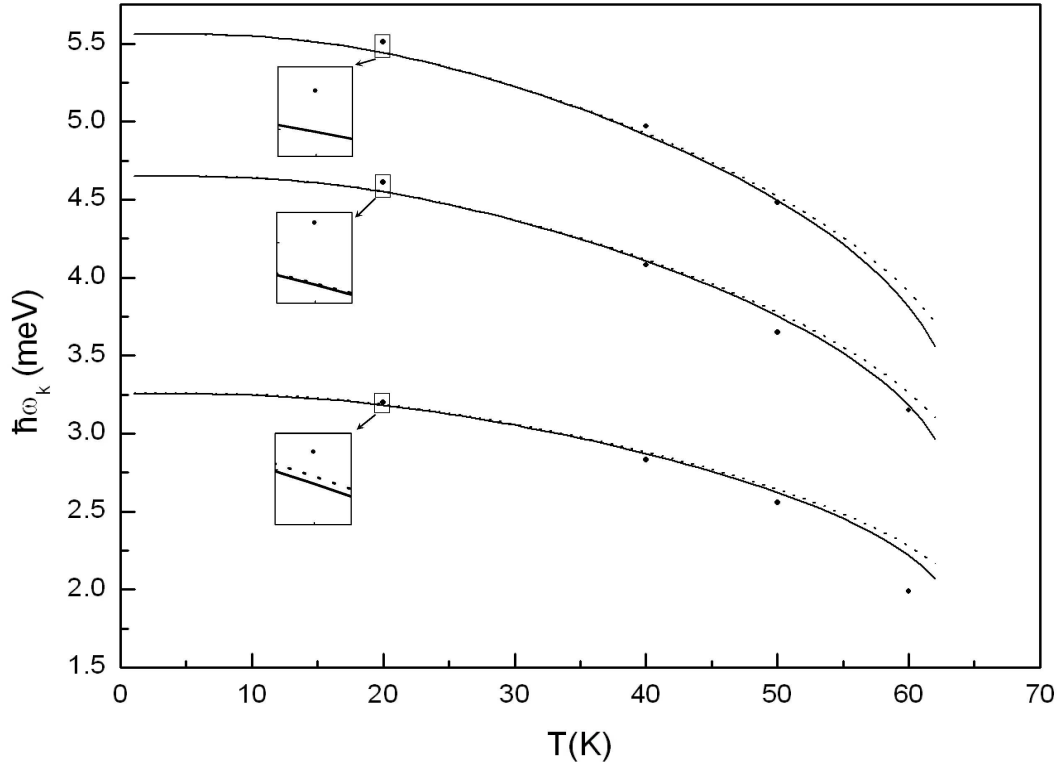


Figure 2: The spin-wave energies in EuO. The dotted lines represent the intermediate-statistics result and the solid lines represent the result of the Holstein-Primakoff representation. The experimental data are taken from [10].

Holstein-Primakoff representation which is a bosonic operator relation with an additional restriction on the occupation number. Since the information of the maximum occupation number is not embodied in the operator relation, the influence of the restriction on the occupation number is not reflected in the result of the spectrum. Consequently, the magnons are approximately treated as bosons in the Holstein-Primakoff treatment: the commutation relation of creation and annihilation operators is taken as the bosonic commutation relation and the statistical distribution is taken as the Bose-Einstein distribution.

In this paper, we construct an intermediate-statistics operator realization in which the information of the maximum occupation number which is equal to an integer n is embodied in the commutation relation of creation and annihilation operators rather than putting in a restriction on the occupation number by hand. Then, from the operator relations, we directly deduce the corresponding statistical distribution function, which is a modified Bose-Einstein statistical distribution and will return to the Bose-Einstein distribution when taking the maximum occupation number n to be ∞ .

It is the starting point that there is a natural relation between the angular momentum and the intermediate statistics with a given maximum occupation number. For the intermediate statistics with a maximum occupation number n , there are $n + 1$ states, $|0\rangle, |1\rangle, |2\rangle, \dots, |n\rangle$. For the angular momentum S , there are $2S + 1$ states $|-S\rangle, |-S + 1\rangle, \dots, |S - 1\rangle, |S\rangle$. This naturally leads us to relate the $n + 1$ states, $|0\rangle, |1\rangle, \dots, |n\rangle$, to the $2S + 1$ angular momentum states, $|-S\rangle, \dots, |S\rangle$. Consequently, we have the relation $n + 1 = 2S + 1$, and then $n = 2S$. From this, we can construct an intermediate-statistics realization and reveal that the statistics of magnons is intermediate statistics.

Based on the results of the intermediate statistics provided in sections 2 and 3, we calculate the dispersion relation of the ferromagnetic spin wave up to the quartic contribution, in which the influence of the interaction between magnons is taken into account, and the dispersion relation of the antiferromagnetic spin wave up to the quadratic contribution. Compared to the result of the Holstein-Primakoff representation, the bosonic operator relation is replaced by the intermediate-statistics operator relation, and the Bose-Einstein distribution is replaced by the intermediate-statistics distribution, so the influence of the restriction on the occupation number is naturally taken into account. Moreover, we also take into account the next-to-nearest-neighbor contribution and the influence of the classical dipolar interaction which is caused by the interaction between the magnetic moments in the ferromagnetic case.

Magnons obey a kind of intermediate statistics. As comparisons, we discuss the relations among the intermediate statistics obeyed by magnons, Haldane-Wu fractional statistics, and the fractional statistics of anyons. The relations among the three intermediate-statistics parameters are given.

For discussing the validity of our result, we compare our result with the exact solution of the one-dimensional spin 1/2 Heisenberg model obtained by the Bethe-ansatz method.

Our results of the dispersion relation of the magnetic systems are based on intermediate

statistics, in which the maximum occupation number is an integer n equaling $2S$. We compare our result with the result by the Holstein-Primakoff representation in which magnons are assumed to obey Bose-Einstein statistics and with the experimental data of *EuO*. The result compares well with the experiment.

In a word, the elementary excitation of the Heisenberg magnetic system obeys a kind of intermediate statistics with a finite maximum occupation number $n = 2S$ rather than Bose-Einstein statistics.

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References

- [1] Grosso G and Parravicini G P, 2000 *Solid State Physics* (San Diego: Academic press)
- [2] Holstein T and Primakoff H, 1940, *Phys. Rev.* **58** 1098
- [3] Dai W-S and Xie M, 2004 *Physica A* **331** 497
- [4] Shen Y, Dai W-S and Xie M, 2007 *Phys. Rev. A* **75** 042111
- [5] Haldane F D M, 1991 *Phys. Rev. Lett.* **67** 937; Wu Y -S, 1994 *Phys. Rev. Lett.* **73** 922
- [6] Wilczek F, 1982 *Phys. Rev. Lett.* **48** 1144; Wilczek F, 1982 *Phys. Rev. Lett.* **49** 957
- [7] Karbach M and Müller G, 1997 *Comput. Phys.* **11** 36
- [8] Cini M, 2007 *Topics and Methods in Condensed Matter Theory: From Basic Quantum Mechanics to the Frontiers of Research* (Berlin: Springer-Verlag).
- [9] Sólyom J, 2007 *Fundamentals of the Physics of Solids, Vol. I, Structure and Dynamics* (Berlin: Springer-Verlag)
- [10] Glinka C J, Minkiewicz V J and Passell L, 1973 *AIP Conf. Proc.* **18** 1060
- [11] Schwinger J, 1965, in *Quantum Theory of Angular Momentum*, ed Biedenharn L (New York: Academic Press) P229–279
- [12] Ambjørn J, Karakhanyan D, Mirumyan M and Sedrakyan A, 2001 *Nucl. Phys. B* **599** 547
- [13] Ruan D, 2003 *Phys. Lett. A* **319** 122
- [14] Martínez-y-Romero R P, Salas-Brito A L and Saldana-Vega J, 1999 *J. Math. Phys.* **40** 2324

- [15] Karakhanyana D and Khachatryan Sh, 2005 *Lett. Math. Phys.* **72** 83
- [16] Sun C-P and Fu H-C, 1989 *J. Phys. A: Math. Gen.* **22** L983; Sun C-P and Ge M-L, 1991 *J. Math. Phys.* **32** 597
- [17] Syromyatnikov A V, 2006 *Phys. Rev. B* **74** 014435
- [18] Wieser R, Vedmedenko E Y and Wiesendanger R, 2008 *Phys. Rev. Lett.* **101** 177202
- [19] Jacak W, Krasnyj J, Jacak L and Kaim S D, 2007 *Phys. Rev. B* **76** 165208
- [20] Bedell K S and Dahal H P, 2006 *Phys. Rev. Lett.* **97** 047204
- [21] Zhang L, Wang J-S and Li B, 2008 *Phys. Rev. B* **78** 144416
- [22] Yamamoto S, 2004 *Phys. Rev. B* **69** 064426
- [23] Zvonarev M B, Cheianov V V and Giamarchi T, 2007 *Phys. Rev. Lett.* **99** 240404
- [24] Nguyesn T M and Cottam M G, 2006 *Surf. Sci.* **600** 4151
- [25] Gao C L, Ernst A, Fischer G, Hergert W, Bruno P, Wulfhkel W and Kirschner J, 2008 *Phys. Rev. Lett.* **101** 167201
- [26] Zhao J *et al*, 2008 *Phys. Rev. Lett.* **101** 167203
- [27] Timm C and Jensen P J, 2000 *Phys. Rev. B* **62** 5634; Yamamoto S and Funase K-I, 2005 *Low Temp. Phys.* **31** 740; Vidal J and Dusuel S, 2006 *Europhys. Lett.* **74** 817
- [28] Bytsko A G, 2005 *J. Math. Sci. N.Y.* **125** 136
- [29] Gentile G, 1940 *Nuovo Cimento* **17** 493
- [30] Dai W-S and Xie M, 2004 *Ann. Phys. (N.Y.)* **309** 295
- [31] Majlis N, 2000 *The Quantum Theory of Magnetism* (Singapore: World Scientific)
- [32] White R M, 2007 *Quantum Theory of Magnetism: Magnetic Properties of Materials*, 3rd ed. (Berlin: Springer-Verlag)
- [33] Khare A, 1997 *Fractional Statistics and Quantum Theory* (Singapore: World Scientific)
- [34] Comtet A, Georgelin Y and Ouvry S, 1989 *J. Phys. A* **22** 3917
- [35] Passell L, Dietrich O W and Als-Nielsen J, 1976 *Phys. Rev. B* **14** 4923